NSG-269-62

10 p.

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A SPHERICAL HARMONIC EXPANSION

N65-88786

~~X63 15527~~

CODE-2A

(NASA CR 51344)

(NASA Grant NsG 269-62)

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8180002

August 6, 1963 10 p Submitted for Publication

To be submitted to

JOURNAL OF GEOPHYSICAL RESEARCH

PERTURBATION OF THE GEOMAGNETIC FIELD - A SPHERICAL HARMONIC EXPANSION

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ABSTRACT

The spherical harmonic expansion of the perturbation of the geomagnetic field is calculated using the magnetopause shape and current system determined previously by the moment technique. The expansion coefficients are compared to those determined by another method by Mead.

Introduction

The work reported in this paper is a direct outgrowth of the calculations of Midgley and Davis (1963) and that paper (hereafter referred to as I) must be read first by anyone who desires to follow the detailed calculations or estimate the accuracy to be expected of the result.

In general, paper I obtains a numerical solution (by a method involving no approximation in the basic equations) to the problem of a dipole normal to a cold, field-free plasma wind. A bounding surface is determined for the field, and the surface currents which properly balance the plasma pressure at each point of the surface and approximately cancel the field everywhere outside the surface are calculated. The magnetic field of these surface currents is calculated at a number of discrete points and field plots made from these values.

It has been brought to the author's attention by Dr. Gilbert Mead that the more useful and traditional way of specifying the geomagnetic field is by means of the coefficients in a spherical harmonic expansion of its scalar potential. These have been obtained by a small modification of the programs used in paper I.

Calculation of the Expansion Coefficients

The scalar potential (defined so that $\underset{\sim}{B} = -\nabla\phi$) of the part of the geomagnetic field due to the surface currents may be expressed as:

$$\phi = R_N J_o \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=0}^{1} \bar{S}^p_{nm} \, \bar{D}^p_{nm} . \tag{1}$$

Following the convention of paper I, R_N is the unit of length (chosen as the distance from the neutral point to the earth-sun axis) and J_o is the unit of current-per-unit-width (chosen as $(N_o M_t U_o^2/\pi)^{1/2}$ where $N_o M_t$ is the density and U_o the velocity of the wind). Thus the coefficients $\bar{S}^p_{nm}$ are dimensionless constants and the $\bar{D}^p_{nm}$ are dimensionless functions of (r,θ,ϕ). The $\bar{D}^p_{nm}$, however, are the solutions of Laplace's equation which vanish at the origin:

$$\bar{D}^p_{nm} = r^n \bar{P}^m_n(\cos\,\theta)\,\cos\,(m\phi - p\tfrac{\pi}{2}) \tag{2}$$

while the D^p_{nm} of paper I (equation 2.4) were the ones vanishing at infinity. Further, in deference to accepted convention, the Schmidt normalized Legendre polynomials (denoted $\bar{P}^m_n$ here) will be used rather than the P^m_n of (2.4). They are defined (Chapman and Bartels, 1940, p. 639) as follows: $\bar{P}^m_n = [(2-\delta_{mo})\,(n-m)!/(n+m)!]^{1/2}\,P^m_n$. The coordinate system here has its z axis pointing toward the sun and its y axis along the dipole. As before, it is actually the vector potential

$$\underset{\sim}{A} = R_N J_o \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{p=0}^{1} \left[\bar{X}^p_{nm} \underset{\sim}{e}_x + \bar{Y}^p_{nm} \underset{\sim}{e}_y + \bar{Z}^p_{nm} \underset{\sim}{e}_z \right] \bar{D}^p_{nm} \tag{3}$$

that is most easily calculated, and so the relationships between the $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and the $\bar{S}$ must be derived. These are:

$$\bar{S}^p_{nm} = (2p-1)\bar{Z}^{1-p}_{nm} - \left[(1+\delta_{m1})(n-m+1)/(n+m)\right]^{1/2} \left[(2p-1)\bar{X}^{1-p}_{nm-1} - \bar{Y}^p_{nm-1}\right] \quad 1 \leq m \leq n$$

$$\bar{S}^p_{nm} = (1-2p)\bar{Z}^{1-p}_{nm} - \left[(1-0.5\delta_{mo})(n+m+1)/(n-m)\right]^{1/2} \left[(2p-1)\bar{X}^{1-p}_{nm+1} + \bar{Y}^p_{nm+1}\right] \quad 0 \leq m \leq n-1 \tag{4}$$

which, of course, differ from (2.8) because $\bar{D}^p_{nm}$ differs from D^p_{nm}.

The $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ are given by (2.12) with some minor changes.

$$\bar{X}^p_{nm} = \int_S j_x(\underset{\sim}{r})\bar{P}^m_n(\cos\theta) \cos(m\phi - p\tfrac{\pi}{2})\, r^{-n-1} dS \text{ , etc.} \tag{5}$$

Since these changes have no effect on the symmetry about the $\phi=0$ or $\phi=\frac{\pi}{2}$ planes, equation (4.1) still applies, and the only non-zero $\bar{S}$ are:

$$\bar{S}^1_{nm} = \bar{Z}^o_{nm} - \left[(1+\delta_{m1})(n-m+1)/(n+m)\right]^{1/2} \left[\bar{X}^o_{nm-1} - \bar{Y}^1_{nm-1}\right] \quad m = 1, 3, 5 \ldots n$$

$$\bar{S}^1_{nm} = -\bar{Z}^o_{nm} - \left[(n+m+1)/(n-m)\right]^{1/2} \left[\bar{X}^o_{nm+1} + \bar{Y}^1_{nm+1}\right] \quad m = 1, 3, 5 \ldots n-1 \tag{6}$$

The first relation is used (because it applies when m=n) to obtain integral expressions for the $\bar{S}^1_{nm}$ analogous to (4.6). These integrals were evaluated for the surface calculated in I, giving the values shown in Table 1.

<u>Transformation to Magnetic Coordinates</u>

The natural coordinate system for the above calculations has its polar axis along the earth sun line; but the most useful coordinate system for the results has its polar axis along the earth's dipole axis. The coordinates in this new system will be denoted by (r, θ', ϕ') where θ' is measured from the dipole axis (the y axis in the old (r, θ, ϕ) system) and ϕ' is measured from the midnight meridian. Denote the scalar potential expansion coefficients appropriate to this new coordinate system by T^p_{nk} so that

$$\phi = R_N J_o \sum_{n=1}^{\infty} \sum_{k=0}^{n} \sum_{p=0}^{1} T^p_{nk} \bar{D}^p_{nk} \; (r,\theta',\emptyset) \quad . \tag{7}$$

In order to express the T^p_{nk} as functions of the $\bar{S}^p_{nm}$, it is necessary to be able to express the $\bar{D}^1_{nm}(r,\theta,\emptyset)$, m=1,3,...n, as linear combinations of the $\bar{D}^p_{nk}(r,\theta',\emptyset')$.

$$\bar{D}^1_{nm}(r,\theta,\emptyset) = \sum_{p=0}^{1} \sum_{k=0}^{n} M^{np}_{mk} \bar{D}^p_{nk} \; (r,\theta',\, \emptyset') \quad . \qquad m=1,3,\ldots n \tag{8}$$

The transformation matrices M^{np}_{mk} can be derived from the equations for the transformation of spherical harmonics (see Rose (1957) equations 4.28a, 4.12, 4.13, and III.20)

$$M^{np}_{mk} = \delta_{po}(1-(-1)^{n+k})(1/2)^{n+1} i^{3m+2k-1} \cdot \sum_{s=t}^{n-k} \frac{(-1)^s \left[(2-\delta_{ko})(n-m)!(n+m)!(n-k)!(n+k)!\right]^{1/2}}{(s+k-m)!(n-k-s)!(n+m-s)!s!} \tag{9}$$

where t=m-k unless m<k in which case t=0; and δ_{po}=0 unless p=0 in which case δ_{oo}=1. The numerical values of some of these matrices are given in Table 2. Inserting equation (8) into equation (1) and comparing with (7) we get

$$T^o_{nk} = \sum_{m=1}^{n} M^{no}_{mk} \bar{S}^1_{nm} \quad . \qquad \text{(m odd only)} \tag{10}$$

This is the equation used in deriving Table 3 from Table 1. ¶ The gradient of (7) was calculated (using the coefficients in Table 3) at those points in the equatorial plane where the field had been calculated directly in paper I. It was found that the fields calculated by these two different methods agreed to better than 0.1 per cent at all points within 0.4R_N of the origin. At points within about 30° of the earth-sun line agreement was better than 0.4 per cent out to 0.9 R_N, but it was not as good toward the tail of the cavity. There is about a 2 per cent discrepancy in the values at 0.7R_N in the night hemisphere, rising to 8 per cent at R_N.

Discussion

The scalar potential of the field produced by the surface currents has also been determined by Mead (1963) by a method involving higher order corrections to Beard's solution. Following Chapman & Bartels (1940), he expresses the expansion of the scalar potential in the form

$$\phi = a \sum_{n=1}^{\infty} \sum_{m=0}^{n} g_n^m \left(\frac{R}{a}\right)^n \bar{P}_n^m(\cos\theta') \cos m\phi' \qquad (11)$$

where a is the radius of the earth at the equator. Equating (7) and (11) and using (5.3) from I with S_{11}^1=-7.0030 and r_o=1.0166R_N/a gives the following relationship between T_{nm}^o and g_n^m.

$$g_n^m = \frac{B_e}{7.003} \left(\frac{1.0166}{r_o}\right)^{n+2} T_{nm}^o \qquad (12)$$

where B_e is the magnitude of the earth's dipole field at the equator and r_o is the distance to the subsolar point in units of earth radii. Using this formula, the results in Table 3 have been expressed in Mead's notation (choosing B_e = .31 gauss) and are compared with Mead's results in Table 4.

It is a moot question as to whether the zero temperature solution or the solution with a superimposed 1% isotropic pressure is more meaningful physically, because the former doesn't close at all on the night side and the latter closes too abruptly. Confining our attention, however, to the discrepancy in the results for the two zero-temperature solutions, it appears that it can be explained almost entirely in terms of the differences in the surface shapes determined by the two methods. For instance, defining $\gamma = r_o/R_N$, for Mead's surface γ = 1.08, while for the author's surface γ = 1.0166. If Mead's value were used in (12) the last line of Table 4 would read: -0.240 0.119 -0.014 -0.020. It is easy to see qualitatively why γ should affect these coefficients the way it does. The dipole moment of the surface is roughly proportional to $r_o^2 R_N J_o = r_o^3 J_o/\gamma$. Since this must equal the earth's dipole moment, the field of the earth's dipole at the subsolar point must be proportional to J_o/γ. The total field there must be $4\pi J_o$, so the field there due to the surface currents ($\simeq 2\pi J_o(2-1/\gamma)$) must increase with increasing γ.

There is no clearcut way to decide which of the two shapes (and therefore which set of coefficients) is more accurate, but in all fairness there is one piece of evidence which favors Mead's result. The calculation of (6.3) in paper I indicates that 53% of the field just inside the subsolar point is contributed by the earth's dipole, while for Mead's solution the corresponding result is probably about 47%. With a plane or spherical boundary, the exact percentages are 50% and 33%. A cylindrical box with a centered dipole can give a result greater than 50%, but it must have a length to diameter ratio of almost three in order to do so.

In conclusion, it is gratifying that the two sets of coefficients agree as well as they do, considering their completely different derivations. It is reasonable to say that the discrepancies between them are probably smaller than the errors due to the oversimplification of the original model.

ACKNOWLEDGMENTS

This work was supported by the National Aeronautics and Space Administration under Grant NsG-269-62. The numerical computations were carried out at California Institute of Technology under Grant NsG-151-61.

REFERENCES

Chapman, S., and J. Bartels, Geomagnetism, Clarendon Press, 1940.

Mead, G. S., [Publisher supply title and reference] J. Geophys. Research 68, , 1963.

Midgley, J. E., and L. Davis, Jr., Calculation by a Moment Technique of the Perturbation of the Geomagnetic Field by the Solar Wind, J. Geophys. Research 68, , 1963.

Rose, M. E., Elementary Theory of Angular Momentum, John Wiley & Sons, 1957.

Table 1. Calculated Expansion Coefficients $\bar{S}^1_{nm}$

n=	1	2	3	4	5	6
m = 1	-4.3080	-1.9373	-.1146	.288	.081	-.087
m = 3			.358	.247	.0001	-.107
m = 5					.089	-.067

Table 2. The Transformation Matrices M^{no}_{mk}

	n=1	n=2	n = 3		n = 4		n = 5			n = 6		
k=	0	1	0	2	1	3	0	2	4	1	3	5
m = 1	1	-1	$-\sqrt{\frac{3}{3}}$	$\sqrt{\frac{5}{8}}$	$\sqrt{\frac{9}{16}}$	$-\sqrt{\frac{7}{16}}$	$\sqrt{\frac{30}{128}}$	$-\sqrt{\frac{56}{128}}$	$\sqrt{\frac{42}{128}}$	$-\sqrt{\frac{100}{256}}$	$\sqrt{\frac{90}{256}}$	$-\sqrt{\frac{66}{256}}$
m = 3			$-\sqrt{\frac{5}{8}}$	$-\sqrt{\frac{3}{8}}$	$\sqrt{\frac{7}{16}}$	$\sqrt{\frac{9}{16}}$	$\sqrt{\frac{35}{128}}$	$-\sqrt{\frac{12}{128}}$	$-\sqrt{\frac{81}{128}}$	$-\sqrt{\frac{90}{256}}$	$\sqrt{\frac{1}{256}}$	$\sqrt{\frac{165}{256}}$
m = 5							$\sqrt{\frac{63}{128}}$	$\sqrt{\frac{60}{128}}$	$\sqrt{\frac{5}{128}}$	$-\sqrt{\frac{66}{256}}$	$-\sqrt{\frac{165}{256}}$	$-\sqrt{\frac{25}{256}}$

Table 3. Transformed Expansion Coefficients T^o_{nk}

n=	1	2	3	4	5	6
k = 0	-4.3080		-.213		.102	
k = 1		1.9373		.379		.152
k = 2			-.310		.007	
k = 3				-.005		.005
k = 4					.064	
k = 5						-.021

Table 4. Comparison of Mead's Expansion Coefficients with the Corresponding T^{o}_{nm} (See equation 12)

	$r_o^3 g_1^o$	$r_o^4 g_2^1$	$r_o^5 g_3^o$	$r_o^5 g_3^2$
Mead - 1% isotropic pressure	-0.277	0.108	-0.012	-0.024
Mead - zero temperature	-0.243	0.121	-0.014	-0.023
Midgley - zero temperature	-0.200	0.092	-0.010	-0.015